UK INTERMEDIATE MATHEMATICAL CHALLENGE

THURSDAY 5th FEBRUARY 2009

Organised by the **United Kingdom Mathematics Trust**
from the School of Mathematics, University of Leeds

http://www.ukmt.org.uk

The Actuarial Profession
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SOLUTIONS LEAFLET

This solutions leaflet for the IMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates.

1. **B** $1 + 2^3 + 4 \times 5 = 1 + 8 + 20 = 29$.

2. **D** The first five non-prime positive integers are 1, 4, 6, 8, 9.

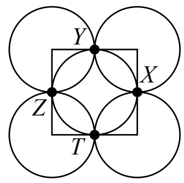
3. **C** The values of these expressions are 5, 8, 9, 8, 5 respectively.

4. **A** The two acute angles in the quadrilateral in the centre of the diagram are both $(180 - 2x)°$ and the two obtuse angles are both $y°$, so $360 - 4x + 2y = 360$. So $y = 2x$.

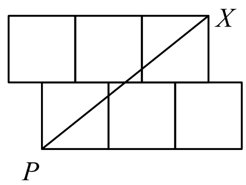
5. **D** Let the number be x. Then $x^2 = 2x^3$, that is $x^2(1 - 2x) = 0$. So $x = 0$ or $x = \frac{1}{2}$. However, x is positive, so the only solution is $x = \frac{1}{2}$.

6. **A** $\frac{4}{5} = \frac{12}{15}$ and $-\frac{2}{3} = -\frac{10}{15}$, so the number half way between these is $\frac{1}{2}\left(\frac{-10}{15} + \frac{12}{15}\right)$, that is $\frac{1}{15}$.

7. **C** As can be seen from the diagram, the square whose vertices are the centres of the original four circles has side of length 2 units and this distance is equal to the diameter of the circle through X, Y, Z and T.



8. **A** The small square on top will be in the upper half of the divided figure. Now consider the figure formed by moving this square to become an extra square on the left of the second row, as shown. It may now be seen from the symmetry of the figure that the line PX splits the new figure in half – with that small square in the upper half. So the line PX does the same for the original figure.

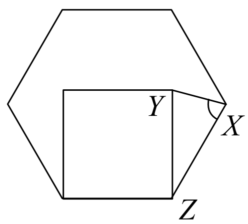


9. **B** The ratio of goats to sheep is 100:155 = 20:31.

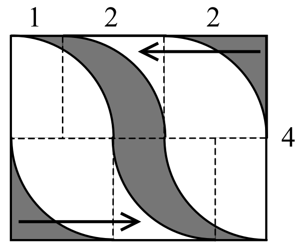
10. **D** There are 51 houses numbered from 100 to 150 inclusive. Of these, 17 are multiples of 3, 11 are multiples of 5 and 4 are multiples of both 3 and 5. So the number of houses Fiona can choose from is $51 - (17 + 11 - 4) = 27$.

11. **E** Note that 2004 is a multiple of 3 (since its digit sum is a multiple of 3) and also a multiple of 4 (since its last two digits form a multiple of 4). So 2004 is a multiple of 12 and hence the part of the pattern between 2007 and 2011 is the same as the part of the pattern between 3 and 7.

12. **D** Let Y and Z be the points shown. The interior angle of a regular hexagon is 120°, so $\angle XZY = 120° - 90° = 30°$. The side of the square has the same length as the side of the regular hexagon, so $YZ = XZ$. Hence triangle XYZ is isosceles and $\angle ZXY = \angle ZYX = \frac{1}{2}(180° - 30°) = 75°$.



13. A If the shaded regions in the top-right and bottom-left corners of the diagram are moved as shown, the area of the shaded region in both the top half and bottom half of the diagram is now that of a 3×2 rectangle which has a quarter of a circle of radius 2 removed from it.



So the total shaded area is
$2\left(3 \times 2 - \frac{1}{4} \times \pi \times 2^2\right) \text{ cm}^2 = \left(12 - 2\pi\right) \text{ cm}^2$.

14. E If n is a positive integer then the units digit of 66^n is 6. So when a power of 66 is divided by 2, the units digit of the quotient is either 3 or 8. Now 66^{66} is clearly a multiple of 4, so $\frac{1}{2}\left(66^{66}\right)$ is even and therefore has units digit 8 rather than 3.

15. B As $\dfrac{1}{x} = 3.5 = \dfrac{7}{2}, x = \dfrac{2}{7}$. So $x + 2 = \dfrac{16}{7}$. Hence $\dfrac{1}{x + 2} = \dfrac{7}{16}$.

16. B If n is an odd prime, then $n^3 + 3$ is an even number greater than 3 and therefore not prime. The only even prime is 2 (which some would say makes it very odd!) and when $n = 2, n^3 + 3 = 11$ which is also prime. So there is exactly one value of n for which n and $n^3 + 3$ are both prime.

17. D Triangles PRS and QPR are similar because: $\angle PSR = \angle QRP$ (since $PR = PS$) and $\angle PRS = \angle QPR$ (since $QP = QR$). Hence $\dfrac{SR}{RP} = \dfrac{RP}{PQ}$, that is $\dfrac{SR}{6} = \dfrac{6}{9}$, that is $SR = 4$.

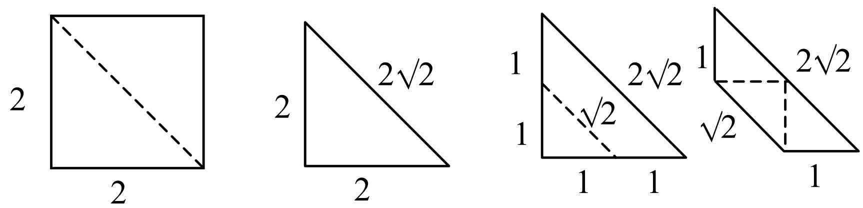
18. B For all positive integer values of p and q, $2p^2q$ and $3pq^2$ have a common factor of pq. They will also have an additional common factor of 2 if $q = 2$ and an additional common factor of 3 if $p = 3$. As the values of p and q are to be chosen from 2, 3 and 5, the largest possible value of the highest common factor will occur when $p = 3$ and $q = 5$. For these values of p and q, $2p^2q$ and $3pq^2$ have values 90 and 225 respectively, giving a highest common factor of 45.

19. E Let the time for which Mary drove at 70 mph be t hours. Then the total distance covered was $\left(55 \times 2 + 70 \times t\right)$ miles. Also, as her average speed over $(2 + t)$ hours was 60 mph, the total distance travelled was $60(2 + t)$ miles.

Therefore $110 + 70t = 120 + 60t$, that is $10t = 10$, that is $t = 1$.

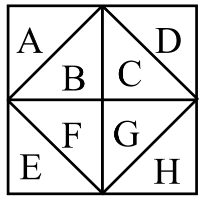
So, in total, Mary's journey took 3 hours.

20. D As can be seen from the figures below, the perimeter of the trapezium is $2 + 3\sqrt{2}$.



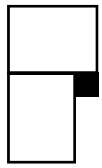
21. E Consider the top row of four dots. One can obtain a triple of dots by eliminating any one of the four – so there are four such triples. The same is true for each of the four rows, each of the four columns and the two main diagonals, giving 40 triples. In addition there are four diagonal lines consisting of exactly three dots, so there are 44 triples in total.

22. D If the first triangle selected to be shaded is a corner triangle, then the final figure will have at least one axis of symmetry provided that the second triangle selected is one of five triangles. For example, if A is chosen first then there will be at least one axis of symmetry in the final figure if the second triangle selected is B, D, E, G or H. The same applies if an inner triangle is selected first: for example, if B is chosen first then there will be at least one axis of symmetry in the final figure if the second triangle selected is A, C, F, G or H.



So, irrespective of which triangle is selected first, the probability that the final figure has at least one axis of symmetry is $\frac{5}{7}$.

23. C Firstly, note that the black squares have side 2 units. The pattern may be considered to be a tessellation of the shape shown on the right. So the ratio of squares to rectangles is 1:2 and hence the fraction coloured black is $\frac{4}{4 + 2 \times 48} = \frac{4}{100} = \frac{1}{25}$.



24. C Reading from the left, we number the statements I, II, III, IV and V.
Statement I is true if and only if $-1 < x < 1$; statement II is true if $x > 1$ or if $x < -1$.
By considering the graph of $y = x - x^2$, which intersects the x-axis at (0, 0) and (1, 0) and has a maximum at $\left(\frac{1}{2}, \frac{1}{4}\right)$, it may be seen that statement V is true if and only if $0 < x < 1$.
We see from the table below that a maximum of three statements may be true at any one time.

	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$x > 1$
True statement(s)	II	none	I, III	none	I, IV, V	none	II

25. E As it is known that the fake coin is heavier than all of the others, it is possible in one comparison to identify which, if any, is the fake in a group of three coins: simply compare any two of the three coins – if they do not balance then the heavier coin is the fake, whereas if they do balance then the third coin is the fake. This means that it is possible to find the fake coin when $N = 9$ using two comparisons: the coins are divided into three groups of three and, using the same reasoning as for three individual coins, the first comparison identifies which group of three coins contains the fake. The second comparison then identifies which of these three coins is the fake. However, it is not possible to identify the fake coin in a group of four coins in one comparison only, so it is not always possible to identify the fake coin using two comparisons when $N = 10$. If less than four are put on each side for the first comparison and they balance, then there are more than three left and the fake coin amongst these cannot be identified in one further comparison. Alternatively, if more than three are put on each side for the first comparison and they do not balance, then the fake coin in the heavier group cannot be identified in one further comparison.